A simple algorithm for Monge-Kantorovich optimal transport

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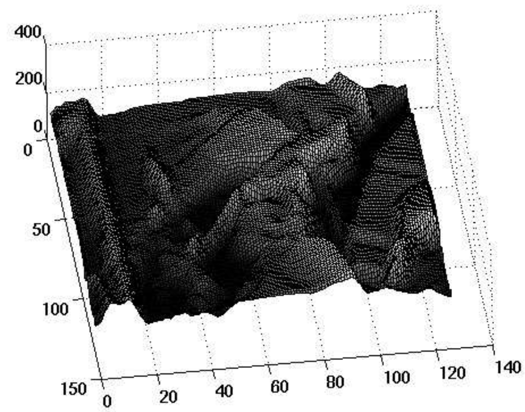
Overview

The Monge-Kantorovich problem introduced by G. Monge [1] in 1781 asks, in its original form, "What is the cheapest way to fill a hole using dirt from a hill of the same volume?" Over the years, important applications of this problem have emerged in such diverse areas as economics, meteorology, astrophysics, probability, and image processing. L. V. Kantorovich [2, 3] won the Nobel Prize in Economics for his work on this and related problems.

Our interest in this problem is in its use to quantify relationships between two images (see [4] for example). This can be done by thinking of the two images in terms of their intensity functions and then by trying to move one intensity landscape to the other. The cheapest way to do this is called the *optimal mapping* or *warp* and the cheapest cost is the *Monge-Kantorovich distance* between the two images. More specifically, we are interested in studying the optimal warps for the information they contain about the differences between the two images. The distance is a single number and is therefore a great simplification of how the images differ. The optimal warp, on the other hand, contains complete information about all the differences between the images. Optimality implies a sort of reduction to the essential differences between the images.

Existing methods for computing the warps are cumbersome, inefficient, and/or nonrobust. This led to our development of a completely new, simple, gradient descent algorithm lacking the drawbacks of the other methods found in the literature.



This image of Lena can be viewed as a function on a rectangle (here of size 128×128). The second plot shows this function as an intensity landscape.

Some details

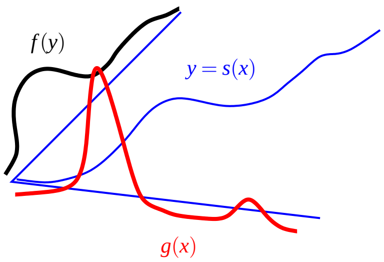
Let $g(x)$ and $f(y)$ be images defined on rectangles R and Q. The optimal Monge-Kantorovich warp of f to g is a mapping s from R to Q that minimizes

$$\int_R |x - s(x)|^2 g(x)\, dx,$$

subject to the constraint that

$$g = (f \circ s) \det(Ds),$$

where Ds is the Jacobian of the mapping. Both the objective function and the constraint are nonlinear and complicated.



We use one dimensional image or intensity profiles to illustrate a warp from one intensity profile to another. The warping function s transforms f to g. More precisely, $g(x) = f(s(x))s'(x)$. The function $y = s(x)$ describes how to move the domain around so that f looks like g. For x's where $ds/dx > 1$ we are compressing the domain of f near $s(x)$ and moving mass towards x; where $ds/dx < 1$ we are stretching the region near $s(x)$ and moving mass away from x.

We have shown [5] that this difficult optimization problem can be replaced with a simpler, *unconstrained* optimization problem: minimize

$$M(u) := \int_R g(x)u(x)\,dx + \int_Q f(y)u^*(y)\,dy$$

over all continuous functions u on R, where u^* is the Legendre-Fenchel transform of u from convex analysis. Furthermore, M is differentiable, and its derivative is

$$M'(u) = g - (f \circ \nabla u)\det(D\nabla u).$$

There is a unique minimizer u, and $s = \nabla u$ will be the optimal Monge-Kantorovich mapping.

Having a simple, explicit expression for $M'(u)$ allows us to compute the minimizer by gradient descent: for any function u, the function $-M'(u)$ points in the direction of fastest decrease of M. Moreover, we can see explicitly that if $M'(u) = 0$, then $s = \nabla u$ satisifies the constraint of the Monge-Kantorovich problem. We thus have an unconstrained optimization problem whose solution solves the much more difficult constrained problem.

Example of an image warp: Lena (left) is warped almost perfectly to Tiffany (right).

Acknowledgements

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